Close Wed: HW_2A, 2B, 2C (5.3,5.4,5.5)

5.5 The Substitution Rule

Entry Task (Motivation):

1. Find the following derivatives

Function	Derivative?
$\cos(x^2)$	
$\sin(x^4)$	
$e^{\tan(x)}$	
$(\ln(x))^3$	
$\ln(x^4+1)$	

2. Rewrite as integrals:

$$\int \qquad dx = \cos(x^2) + C$$

$$\int \qquad dx = \sin(x^4) + C$$

$$\int \qquad dx = e^{\tan(x)} + C$$

$$\int \qquad dx = (\ln(x))^3 + C$$

$$\int \qquad dx = \ln(x^4+1) + C$$

3. Guess and check the answer to:

$$\int 7x^6 \sin(x^7)\, dx =$$

Observations:

1. We are reversing the “chain rule”.
2. In each case, we see
   “inside” = a function inside another
   “outside” = derivative of inside

To help us mechanically see these connections, we use what we call:

The Substitution Rule:
If we write u = g(x) and du = g’(x) dx, then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Aside (you do not need to write this)

Some theory

Recall:

$$\int_a^b f(g(x))g'(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(g(x_i))g'(x_i)\Delta x$$

If we replace u = g(x), then we are "transforming" the problem from one involving *x* and *y* to one with *u* and *y*.

This changes ***everything*** in the set up. The lower bound, upper bound, widths, and integrand all change!

Recall from Math 124 that

$$g'(x) = \frac{du}{dx} \approx \frac{\Delta u}{\Delta x}$$

(with more accuracy when Δx is small)

Thus, we can say that

$$g'(x)\Delta x \approx \Delta u$$

In other words, if the width of the rectangles using *x* and *y* is Δx, then the width of the rectangles using *u* and *y* is $g'(x)\Delta x$.

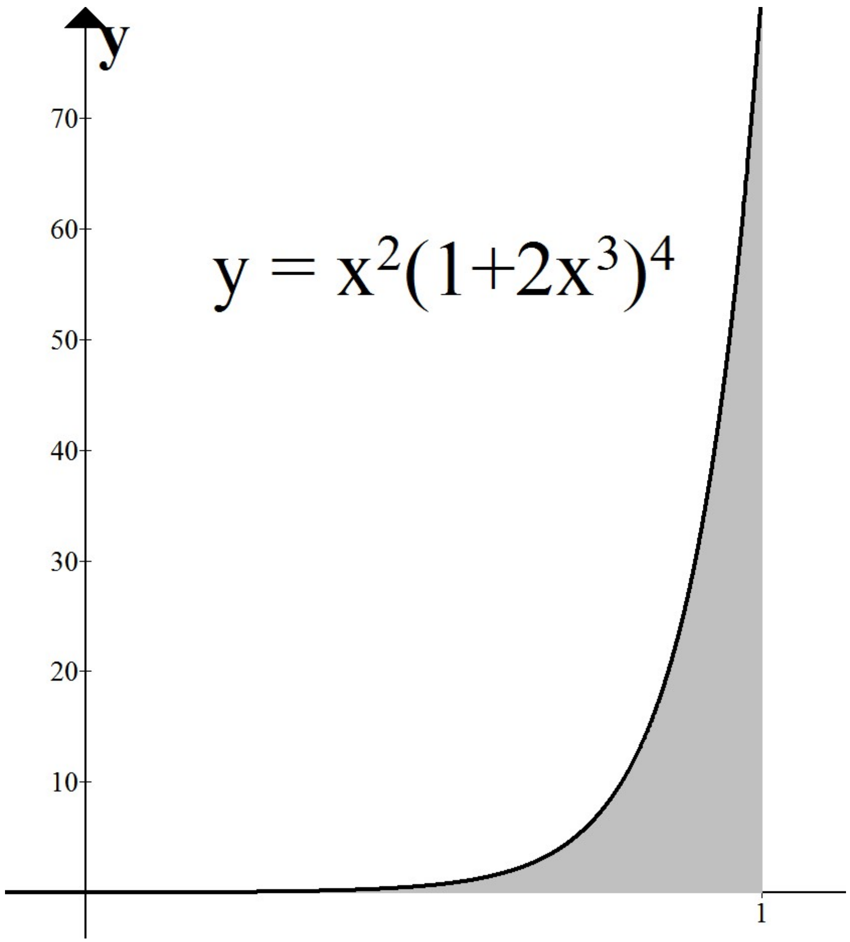
And if we write $u_i = g(x_i)$, then

$$\int_a^b f(g(x))g'(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(g(x_i))g'(x_i)\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} f(u_i)\Delta u$$

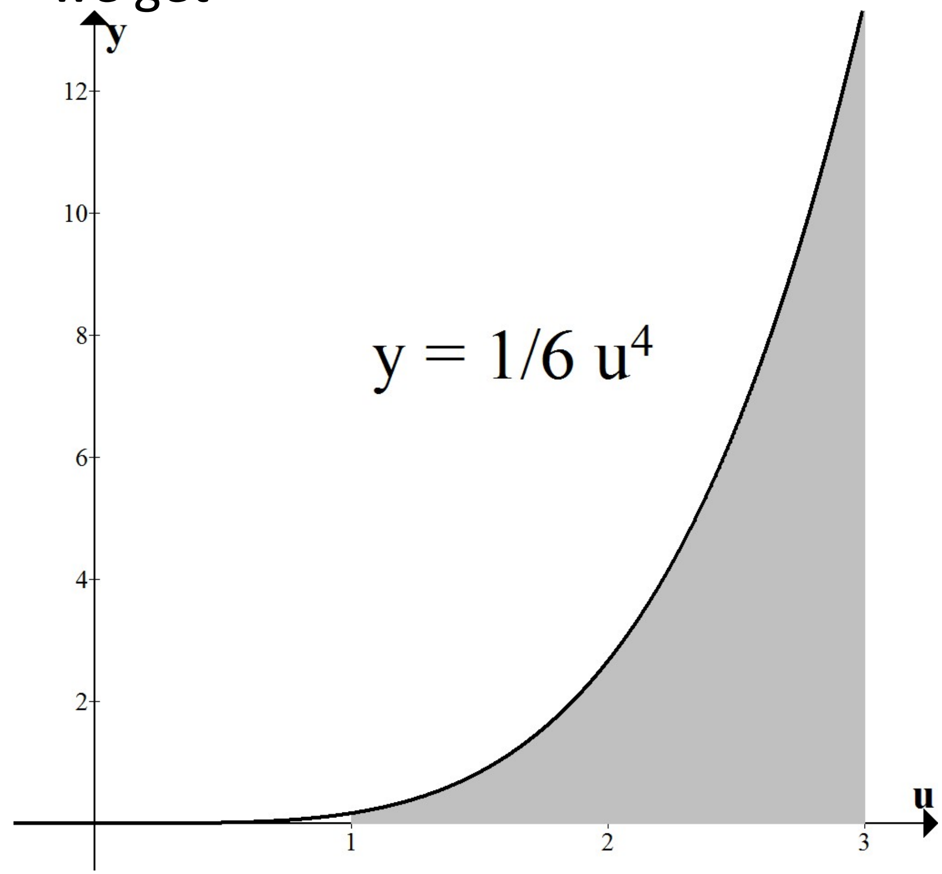
$$= \int_{g(a)}^{g(b)} f(u)du$$

Here is a visual example of this transformation



$$\int_0^1 x^2(1+2x^3)^4 dx$$

Using u = 1 + 2x^3 and du = 6x^2 dx, we get



$$\int_1^3 \frac{1}{6} u^4 du$$

Examples:

First, try u = “inside function”

1. $\int x^4(1+x^5)^{31}\,dx$

2. $\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$

3. $\displaystyle\int_2^3 x^2 e^{x^3}\,dx$

4. $\displaystyle\int \frac{x\sin(x^2)}{\cos^2(\cos(x^2))}\,dx$

Examples:

Then, try u = "denominator"

1. $\int_0^1 \frac{x}{x^2+3}\,dx$

2. $\int \tan(x)\,dx$

What to do when the “old” variable remains:

Examples:

1. $\int x^3\sqrt{2+x^2}\,dx$

2. $\int \frac{\mathrm{x}^7}{x^4+1}\,dx$